A.M. ≥ G.M. , CBS inequality, etc

1. Show that $a^b b^a \le \left(\frac{a+b}{2}\right)^{a+b}$, where $a, b > 0$.

2. Show that $(x^m + y^m)^n < (x^n + y^n)^m$, if $m > n > 0$ and $x, y > 0$.

3. Let $a_1, a_2, \ldots, a_n$ be n real numbers and let p and q be two real numbers.

Prove that if p and q are of the same sign, then

$$n(a_1^{p+q} + a_2^{p+q} + \ldots + a_n^{p+q}) \ge (a_1^p + a_2^p + \ldots + a_n^p)(a_1^q + a_2^q + \ldots + a_n^q)$$

and if p and q are have different signs, then

$$n(a_1^{p+q} + a_2^{p+q} + \ldots + a_n^{p+q}) \le (a_1^p + a_2^p + \ldots + a_n^p)(a_1^q + a_2^q + \ldots + a_n^q).$$

4. Show that $\left(\frac{a+b+c+\ldots+k}{n}\right)^{a+b+c+\ldots+k} \le a^a b^b c^c \ldots k^k$, where $a, b, c, \ldots, k$ are n positive numbers.

5. (Weierstrass inequality) Show that $\prod_{r=1}^{n}(1+x_r) > 1 + \sum_{r=1}^{n} x_r$, where $x_1, x_2, \ldots, x_n > 0$, $n \ge 2$.

6. Let $A + B + C = \pi$ $(A, B, C > 0)$, prove that $\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \ge 1$.

7. Let $A + B + C = \pi$ $(A, B, C > 0)$, prove that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$.

8. Given that $A + B + C = \pi$ $(A, B, C > 0)$, prove that:

(i) $\cos A + \cos B + \cos C \le \frac{3}{2}$ **(ii)** $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \le \frac{3\sqrt{3}}{8}$.

9. (Weighted A.M. ≥ Weighted G.M.)

(a) Show that **(i)** if $a > 0, b > 0$, then $\ln a \le \ln b$ implies $a \le b$.

(ii) if $x > 0$, then $\ln x \ge x - 1$.

(b) Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $t_1, t_2, \ldots, t_n$ be 2 sets of positive numbers such that:

$\alpha_1 t_1 + \alpha_2 t_2 + \ldots + \alpha_n t_n = \alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$. Show that $t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_n^{\alpha_n} \le 1$.

Hence show that $a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n} \le \left(\frac{m_1 a_1 + m_2 a_2 + \ldots + m_n a_n}{m_1 + m_2 + \ldots + m_n}\right)^{m_1+m_2+\ldots+m_n}$.

10. **(i)** If $x > 0$ and p is a positive integer, show that $\frac{x^{p+1} - 1}{p+1} \ge \frac{x^p - 1}{p}$

and that the equality holds only if $x = 1$.

(ii) Let $x_1, x_2, \ldots, x_n$ be positive numbers and $\sum_{i=1}^{n} x_i \ge n$.

(a) Show that, for any positive integer m, $\sum_{i=1}^{n} x_i^m \ge n$

(b) If $\sum_{i=1}^{n} x_i^m = n$ for some integer m greater than one, show that: $x_1 = x_2 = \ldots = x_n = 1$.

(iii) Using **(ii)**, or otherwise, show that for any positive numbers $y_1, y_2, \ldots, y_n$,

$$\frac{y_1^m + y_2^m + \ldots + y_n^m}{m} \ge \left(\frac{y_1 + y_2 + \ldots + y_n}{m}\right)^m,$$

and that the equality holds only when $m = 1$ or $y_1 = y_2 = \ldots = y_n$.